ISSN 2229-5518

On Fractional Derivatives: The Non-integer Order of the Derivative

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Abstract— Integration of significant concepts in mathematics makes it possible to extend the concept of the order of a derivative from an integer order to the non-integer order. This study aimed to show in an exhaustive and systematic manner how an integer ordered-derivative of functions monomial x^p ($p \in \mathbb{Z}$), constant functions, exponential function e^{ax}, and sine and cosine functions of the form $\sin bx$ and $\cos bx$ may be generalized to the positive non-integer order called the fractional derivative. The method used in the study is purely expository and therefore is based on the underlying concepts relevant to the derivative of a function and other topics arising in the process of derivation. The results highlight that in the case of non-integer ordered derivatives, the concept of a zero-derivative for constant functions only apply for integer-ordered derivatives. Additionally, fractional derivative of some monomials x^pmay result to expressions involving complex numbers.

Index Terms— calculus, constant functions, derivative, exponential functions, monomial functions, order of a derivative, trigonometric functions,

1 Introduction

Calculus is one of the fundamental and underlying branches of mathematics. It was known before as inifinitesimal calculus, which is focused on functions, infite series and sequences, limits, derivatives, and anti-derivatives. Calculus has been developed during the later part of 17th century, independently, by two mathematicians Isaac Newton and Gottfried Leibniz. The primary concepts that both of them contributed are formulas on differentiation and integration, higher derivatives, and the idea of approximating polynomials. In the 17th century, European mathematicians discussed the idea of a derivative.

During the late 1695, L'Hopital wrote to Leibniz asking him about a particular notation seen in his publications for the n-th derivative of the linear function $f(x) = x$, $D^n x / Dx^n$. L'Hopital asked Leibniz what would the result be if $n = 1/2$; that is ``What if $n$ is fractional?''. This gave birth to fractional calculus, the thinking when the order $n$ becomes non-integer. ``This is an apparent paradox from which, one day, useful consequences will be drawn'', Leibniz replied, together with ``$D^{1/2}x$ will be equal to $x\sqrt{dx:x}$''. After L'Hopital and Leibniz' first inquisition, fractional calculus has intrigued the minds of the mathematicians. These people proposed different definitions that fit the concept of a non-integer order of derivative or integral, the most popular of which the Riemann-Liouville and Caputo definition. [15]

Fractional calculus is the branch of calculus that generalizes the derivative of a function to non-integer order. The concept of order of derivative of a function is traditionally associated to a positive integer. It is generally known that integer-order derivatives have clear physical and geometric interpretations. On the other hand, the case of fractional-order differentiation which characterizes a rapidly growing field both in theory and in applications to real word problems [6],[7],[8],[9] is continually getting the interests of many mathematicians because of its significant contributions to sciences and engineering.

As mentioned by Loverro, while almost anyone can verify that $x^{3}= x \cdot x \cdot x$, how might one describe the physical meaning of $x^{3.4}$, or moreover the transcendental exponent x^{π}. One cannot conceive what it might be like to multiply a number by itself 3.4 times, and yet, these expressions have actual values for any given x, verifiable using series expansion or by the use of calculator or computer. This is also the case for derivatives and integrations. For example, given $f(x) = -3x^4$, it is considerably easy for one to get its first, second , third or fourth derivative. To illustrate,

$$f(x) = -3x^4$$
$$f^{(1)}(x) = -12x^3$$
$$f^{(2)}(x) = -36x^2$$
$$f^{(3)}(x) = -72x$$
$$f^{(4)}(x) = -72$$

An inquisitive mind might ask what if n, the order, were not restricted to an integer value? Can one actually get the α^{th} derivative of f when $m-1 < \alpha < m$ and $m \in \mathbb{N}$. For instance, given the same $f(x)$, what could be the $f^{3.15}(x)$?

Kleinz and Osler, in their study, gave an exploration of the various approaches to the notion of fractional derivative. Fractional derivatives of exponential functions, trigonometric functions especifically sine and cosine functions, and the monomial x^p, which are the main concerns of this paper were presented in the study. The difference between positive and negative order of differentiation is also given explanation in the study. This study, compared to Kleinz and Osler, will show how the integer order of a derivative is generalized to the non-integer order for the functions included in their study. Along with all these, some contradications on the use of particular definitions mentioned in the paper were presented and resolved as part of the conclusion.

Bologna, in his study entitled Short Introduction to fractional Calculus, used fractional calculus to deal with anomalous diffusion processes. His study showed that fractional cal-

ISSN 2229-5518

culus is a very helpful tool to perform calculation specifically dealing with power law. His paper presented some of the properties of fractional derivative that will be very significant in the course of discussion in this paper. While his study focused more on the application part, this paper, on the other hand, will have the properties as one of its primary interests.

Additionally, the concepts, definitions, and execution of fractional calculus including a discussion of notations, operators, and fractional order differential equations were exposed by Adam Loverro. He also cited how these may be used to solve several modern problems. In Applications of Fractional Calculus, a study conducted by Mehdi Dalir and Majid Bashour, different definitions of fractional derivatives and fractional integrals (Differintegrals) are considered. Using these definitions, explicit formula and graphs of some special functions are derived. Also, applications of the theory of fractional calculus and different fields were reviewed.

Over the years, calculus in the Philippine academe customarily focuses on the positive integer-order when it comes to topics about derivatives (and even integrals) of functions. This is also the case for the differential operators in differential equations. Diverse studies, both in the graduate and the undergraduate, have been unceasingly conducted in the purpose of further exposing calculus and its wide range of applications. Yet, inquest on the idea of fractions as order of the derivative have not much captured the interests of many especially those in the local perspective; or most probably, some may not have realized its existence. The study of fractional calculus is something uncommon and yet absorbing. The term *fractional* derivatives may sound mediocre given the notion of the straightforwardness of the formulas of derivatives. Yet the researcher came upon the mathematical elegance of fractional derivatives. It is one subject which integrates different mathematical concepts one may not even conceive about using of when finding a derivative of a function. Studies on fractional derivatives accessible nowadays, if not all, are mostly very technical and overly profound, something that one may not easily grasp. The researcher aspires through this study to adequately exhibit the content of fractional derivatives in a less rigorous manner, a way that is considerate and practical to a reader. In that way, the topic will not be viewed as unfriendly one because of its complexities. It is for these reasons which the researcher holds significant that aspire her to conduct this study and explore the subject.

The paper seeks to present a detailed study on fractional derivatives. Specifically, the study aims to generalize integer order to the non-integer order derivatives of functions as (a) monomial x^p, $p \in \mathbf{z}$; (b) constant functions; (c) exponential function e^{ax}; (d) sine and cosine functions of the form $\sin bx$ and $\cos bx$.

This paper shows a generalization of the existing derivative formulas for a non-integer order of the derivative. It will consolidate, connect, and generalize the different properties of fractional derivatives for different types of functions. These properties will be used to arrive at the formulas for the fractional derivatives of different types of functions. This study will be beneficial primarily to mathematics enthusiasts -- teachers, students, and researchers, as this will give an informative and analytical discussion about the topic which they can use as basis or reference for further advanced studies. This study will make them aware that generally, not all formulas for finding the derivatives of functions are straighforward as practiced in differential calculus. By getting to know the whole process of fractional differentiation, they will be able to apply and connect their knowledge on iterated integrals, improper integrals, gamma functions, etc. just to obtain a fractional derivative of a function. In addition, for the teachers in the tertiary or graduate school, because of the detailed and easy-to-understand approach that this paper will give especially in obtaining a function's fractional derivative, they may integrate this as a supplementary topic in their mathematical analysis or advanced calculus course syllabus. Lastly, for the students, they will be adept in extending their knowledge in obtaining the derivative of a function from positive integer-order to fractional order. This study will give them the mark on the beauty and unboundedness of the derivative. This may give a new and intriguing impression about the derivative of a function compared to the traditional method of evaluating it.

2 Methodology

The paper makes use of the expository method of research. Journals, books and electronic references are the main resources for the conduct of the study. The researcher read, analyzed, and scrutinized a number of related studies on fractional derivatives to come about the desired objectives. Comparative study on the different focus of different researches has been done in order to consolidate the ideas which will be presented in this paper. Furthermore, consultation with the experts on the field were sought to widen and deepen the understanding and presentation of all the information.

In view of the objective, topics presented in the foregoing sections of this paper will be used as the mathematical foundations to attain such objective. As mentioned, formulas for the derivatives of commonly used functions such as monomial x^p, $p \in \mathbf{z}$, constant functions, exponential function e^{ax}, and the sine and cosine functions of the form $\sin bx$ and $\cos bx$ will be derived. However, the formulas for fractional derivatives that will be discussed in this paper will cover only those whose integrals may be evaluated from 0 to x, that is $\int_0^x f(t)dt$. Although the study of fractional derivatives also encompasses that of the *negative* non-integer order derivative, this paper will focus only on fractional order α, where $m-1 < \alpha < m$ and $m \in \mathbf{n}$.

3 Fractional Derivatives

3.1 Monomial x^p

Beginning with the derivative of the monomial x^p, consider the function $y = x^p$, $p \in \mathbf{z}^+$.

ISSN 2229-5518

$$\begin{aligned} D_x y &= p x^{p-1} \\ D_x^2 y &= p(p-1)x^{p-2} \\ D_x^3 y &= p(p-1)(p-2)x^{p-3} \\ &\vdots \\ D_x^n y &= \frac{p!}{(p-m)!} x^{p-n} \end{aligned}$$

Replacing n by an arbitrary $\alpha \geq 0$ and the factorial by gamma,

$$D_x^\alpha y = D_x^\alpha x^p = \frac{\Gamma(p+1)}{p+1-\alpha} x^{p-\alpha}, \qquad p \in \mathbf{z}^+$$

In the case for which the power of x is negative, cosider the function $y = x^{-p}, p \in \mathbf{z}^+$,

$$\begin{aligned} D_x y &= -p x^{-p-1} \\ D_x^2 y &= -p(-p-1)x^{-p-2} \\ &= (-1)^2 p(p+1)x^{-(p+2)} \\ &= p(p+1)x^{-(p+2)} \\ D_x^3 y &= -p[(-(p+1)][(-(p+2)]x^{-p-3} \\ &= (-1)^3 p(p+1)(p+2)x^{-(p+3)} \\ &\vdots \\ D_x^n y &= -p[(-(p+1)][(-(p+2)] \cdots (-p-n+1)x^{-p-n} \\ &= (-1)^n p(p+1)(p+2) \cdots (p+n-1)x^{-(p+n)} \\ &= (-1)^n \frac{(p+n-1)!}{(p-1)!} x^{-(p+n)} \end{aligned}$$

Generalizing the above with n as any arbitrary $\alpha \geq 0$ and the factorial by gamma,

$$D_x^\alpha y = D_x^\alpha x^{-p} = (-1)^\alpha \frac{\Gamma(p+\alpha)}{p+\alpha-\alpha} x^{-(p+\alpha)}, \qquad p \in \mathbf{z}^+$$

and since $e^{i\pi} = \cos\pi + i\sin\pi = -1$, then

$$D_x^\alpha x^{-p} = e^{i\pi\alpha} \frac{\Gamma(p+\alpha)}{p+\alpha-\alpha} x^{-(p+\alpha)}, \qquad p \in \mathbf{z}^+$$

3.2 Constant Functions

To illustrate how the fractional derivative of a constant is derived, the previously generalized formula of fractional derivative of monomial x^p is used. From

$$D_x^\alpha x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha},$$

if $y = 1$ then it can be expressed as

$$\begin{aligned} D_x^\alpha 1 = D_x^\alpha x^0 &= \frac{\Gamma(0+1)}{\Gamma(0+1-\alpha)} x^{0-\alpha}, \\ &= \frac{\Gamma(1)x^{-\alpha}}{\Gamma(1-\alpha)} \\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \end{aligned}$$

It is interesting to note that the derivative cannot be 0 considering the acceptable values for α or the gamma. Applying a property of the fractional derivative,

$$\begin{aligned} D_x^\alpha C &= D_x^\alpha C \cdot x^0 \\ &= C \cdot D_x^\alpha x^0 \\ &= C \cdot \frac{x^{-a}}{\Gamma(1-\alpha)} \\ &= \frac{C}{x^a \Gamma(1-\alpha)} \end{aligned}$$

This is 0 only if $\alpha \in \mathbf{n}$ from the relationship

$$\Gamma(-z) = \frac{-\pi}{z\Gamma(z)\sin\pi z}$$

Thus all the integer derivative of a constant is zero, but not the non-integer derivatives.

3.3 Exponential Function e^{ax}

Generalizing the derivative of exponential function of the form e^{ax} from the integer order, the limit definition of the derivative will be used.

$$D_x f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Iteration of this equation to the n^{th} derivative will give

$$D_x^n f(x) = \lim_{h\to 0} h^{-n} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x + (n-m)h)$$

In the above equation, with $f(x) = e^{ax}$,

$$\begin{aligned} D_x^\alpha e^{ax} &= \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha}{k} e^{a(x+(\alpha-k)h)} \\ &= \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha}{k} e^{ax} (e^{ah})^{\alpha-k} \\ &= e^{ax} \lim_{h\to 0} h^{-\alpha} \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha}{k} (e^{ah})^{\alpha-k} \\ &= e^{ax} \lim_{h\to 0} h^{-\alpha} \textstyle\sum_{k=0}^{\alpha} \binom{\alpha}{k} (e^{ah})^{\alpha-k} (-1)^k \\ &= e^{ax} \lim_{h\to 0} h^{-\alpha} (e^{ah} - 1)^\alpha \end{aligned}$$

$$= e^{ax} \lim_{h\to 0}\left(\frac{e^{ah}-1}{h}\right)^{\alpha}$$
$$= e^{ax} \lim_{h\to 0}\left(\frac{ae^{ah}}{1}\right)^{\alpha}$$
$$= e^{ax} a^{\alpha}$$
$$D_x^{\alpha} e^{ax} = a^{\alpha} e^{ax}$$

It is worth noting that with the preceeding derivation, the characteristic property of the n^{th} derivative of the exponential function e^{ax} is preserved for any arbitrary n.

3.4 Sine and Cosine Functions of the form $\sin bx$ and $\cos bx$

The fractional derivatives of these trigonometric functions will be generalized using the preceding formula and essential concepts in complex analysis. Extending α and taking it to be any complex number bi,

$$D_x^{\alpha} e^{ibx} = (ib)e^{ibx} = i^{\alpha} b^{\alpha} e^{ibx} \tag{1}$$

Since by De Moivre's formula,

$$i^{\alpha} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\alpha} = \cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}$$

and with Euler's formula, in "(1)"

$$D_x^{\alpha} e^{ibx} = b^{\alpha}\left(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}\right)(\cos bx + i\sin bx) \tag{2}$$

Also, note that

$$D_x^{\alpha} e^{ibx} = D_x^{\alpha}(\cos bx + i\sin bx)$$

and by property of differential operator,

$$D_x^{\alpha} e^{ibx} = D_x^{\alpha}\cos bx + i\,D_x^{\alpha}\sin bx \tag{3}$$

Equating "(2)" and "(3)",

$$D_x^{\alpha}\cos bx + i\,D_x^{\alpha}\sin bx = b^{\alpha}\left(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}\right)(\cos bx + i\sin bx)$$
$$= b^{\alpha}\left(\cos\frac{\alpha\pi}{2}\cos bx - \sin\frac{\alpha\pi}{2}\sin bx\right)$$
$$+ib^{\alpha}\left(\sin\frac{\alpha\pi}{2}\cos bx + \cos\frac{\alpha\pi}{2}\sin bx\right)$$
$$= b^{\alpha}\left[\cos\left(\frac{\alpha\pi}{2} + bx\right) + i\sin\left(\frac{\alpha\pi}{2} + bx\right)\right]$$
$$= b^{\alpha}\cos\left(\frac{\alpha\pi}{2} + bx\right) + ib^{\alpha}\sin\left(\frac{\alpha\pi}{2} + bx\right)$$

Equating the real and imaginary parts,

$$D_x^{\alpha}\cos bx = b^{\alpha}\cos\left(bx + \frac{\alpha\pi}{2}\right)$$

$$D_x^{\alpha}\sin bx = b^{\alpha}\sin\left(bx + \frac{\alpha\pi}{2}\right)$$

CONCLUSION

Based on the foregoing discussions, the researcher gives the following findings and conclusions.

1. Fractional derivatives of indicated types of functions are as follows:
   - Monomial x^p

$$D_x^{\alpha} y = D_x^{\alpha} x^{-p} = (-1)^{\alpha}\frac{\Gamma(p+\alpha)}{p+\alpha-\alpha}x^{-(p+\alpha)}, \qquad p \in \mathbf{z}^+$$
$$D_x^{\alpha} x^{-p} = e^{i\pi\alpha}\frac{\Gamma(p+\alpha)}{p+\alpha-\alpha}x^{-(p+\alpha)}, \qquad p \in \mathbf{z}^+$$

   - Constant Functions

$$D_x^{\alpha} C = \frac{C}{x^{\alpha}\Gamma(1-\alpha)}$$

   - Exponential Function e^{ax}

$$D_x^{\alpha} e^{ax} = a^{\alpha} e^{ax}$$

   - Trigonometric Functions $\sin bx$ and $\cos bx$

$$D_x^{\alpha}\cos bx = b^{\alpha}\cos\left(bx + \frac{\alpha\pi}{2}\right)$$
$$D_x^{\alpha}\sin bx = b^{\alpha}\sin\left(bx + \frac{\alpha\pi}{2}\right)$$

2. Unlike in the integer-ordered derivative, fractional derivative of a monomial x^{-p} where $p \in \mathbf{z}$ may result to a complex number.
3. The fractional derivative of a constant is not zero while the integer ordered derivative of a constant is zero.

ACKNOWLEDGMENT

The author wishes to thank K.G.Guce, F.V.Reyes, Ph.D., N.A.Kalalo, and T.A.Kalalo, and A.A.Macalalad. This work was supported in part by a grant from Fund for Assistance to Private Education and De La Salle Lipa, Philippines.

REFERENCES

[1] R. Larson and B. Edwards, *Calculus*. Belmont, Calif.: Cengage Learning, pp. 95-149, 2012.

[2] J. Stewart, *Calculus: Early Transcendentals, 6th Ed.* Belmont, Calif.: Thomson Brooks/Cole, pp. 172-180, 2008.

[3] R. Minton and R. Smith, *Calculus: Concepts and Connections*. USA: McGraw Hill, 2006

[4] J. Brown and R. Churchill, *Complex Variables and Applications, 8th Ed*. USA: McGraw Hill, pp. 16-24, 2004

[5] K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York, USA: John Wiley & Sons, Inc. 1993

[6] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. Italy: Imperial College Press, University of Bologna, pp. 1-22, 2010

[7] P. Butzer and U. Westphal, "An Introduction to Fractional Calculus. In Applications of Fractional Calculus in Physics," *World Scientific Publication, Co.,* pp. 1-85, 2000

ISSN 2229-5518

[8] Z. Avazzadeh, B. Shafiee and G. Loghmani, "Fractional Calculus for Solving Abel's Integral Equations Using Chebyshev Polynomial," *Applied Mathematical Sciences*, Vol. 5, No. 45, pp. 2207-2216, 2011

[9] M. Dalir and M. Bashour, "Applications of Fractional Calculus," *Applied Mathematical Sciences*, Vol. 4, No. 21, pp. 1021-1032, 2010

[10] V. Tarasov, "Fractional Derivative as Fractional Power of Derivative," *International Journal of Mathematics*,Vol. 18, No. 3, pp. 281-299, 2007

[11] D. Bhatta, "Few Fractional Order Derivatives and Their Computations," *International Journal of Mathematics*, Vol. 38, No. 4, pp. 449-460, June 2007

[12] M. Kleinz and T. Osler, "A Child's Garden of Fractional Derivatives," *The College Mathematics Journal*, Vol. 31, No. 2, pp. 82-88, March 2000

[13] S. Tu, "Commutativity of Leibniz Rules in Fractional Calculus," *Pergamon: Computers and Mathematics with Applications*, Vol. 40, pp. 303-312, 2000

[14] M. Bologna, "Short Introduction Fractional Calculus," unpublished.

[15] A. Lovero, "Fractional Calculus: History, Definitions and Applications for the Engineer," unpublished.